manuscripta math. 82, 393 - 406 (1994)

manuscripta mathematica
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3-FOLDS IN $\mathbb{P}^5$ OF DEGREE 12

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Let $\hat{X}$ be a 3-dimensional submanifold of $\mathbb{P}^5$ of degree 12. This article gives, up to one case, a complete classification of the deformation classes of those 3-folds. The main tools used are methods already applied in the classification of degrees 9 to 11 and adjunction theoretic results. We show here how the 2^{nd} reduction of $\hat{X}$ can be applied to analyze the birational structure of $\hat{X}$ or even exclude the existence of $\hat{X}$.

INTRODUCTION

Submanifolds of $\mathbb{P}^5$ of codimension 2 are 3-folds of special interest because there exist non-complete intersection examples, even examples of Kodaira-dimension $-\infty$. This is interesting in view of the Hartshorne-conjecture which says that submanifolds of $\mathbb{P}^N$ with dimension bigger than $\frac{2}{3}N$ should be complete intersections. Finding examples of 3-folds in $\mathbb{P}^5$ with Kodaira-dimension smaller than 3 is also attractive because we know that the degree of those 3-folds is bounded [BOSS 1].

A 3-dimensional manifold can be embedded in general only into $\mathbb{P}^7$, whence 3-folds in $\mathbb{P}^5$ have to fulfil a lot of further restrictive conditions. This allowed a complete classification up to degree 8 [O1],[O2],[I1],[I2],[I3], continued by Beltrametti, Schneider and Sommese [BSS 1],[BSS 2] for degrees 9 to 11. This article, which is a summary of the author's dissertation, gives a brief outline of the degree-12 case. A complete classification is given up to one set of invariants for which neither an example can be given nor can the non-existence be shown.

The main tool for the classification in degree 12 are methods already used by [BSS 1] and [BSS 2] in degrees 9 to 11 and a systematic study of the 2^{nd} reduction. Finding relations between the invariants of a 3-fold $\hat{X} \subset \mathbb{P}^5$ and those of its 2^{nd} reduction $(X, \mathcal{K})$ plays a very important role. The necessary computations are done in detail in [E] and [BSS 3]. In fact, I was informed by Sommese about the existence of certain formulae relating the invariants of $\hat{X}$ and X the proof of which I worked out independently.

As to finding examples of 3-folds in $\mathbb{P}^5$ of degree 12 one can use the well-known liaison-techniques. Yet 2 examples have to be constructed in a different way. In one example described here in a way originating to Schreyer we use the computer algebra program *Macaulay* to show the smoothness.
The content of this article is the following

Theorem 1. *Each 3-fold in $\mathbb{P}^5$ of degree 12 belongs to one of the following deformation classes. (Refer to paragraph* 1 *for a definition of the invariants.)*

case	$\hat{d}_0$	$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_3$	$g(\hat{X})$	$\chi(\mathcal{O}_{\hat{X}})$	$\chi(\mathcal{O}_{\hat{S}})$	$e(\hat{X})$	$\kappa(\hat{X})$	d_0	d_1	*type*
1	*12*	*16*	*8*	*0*	*15*	*2*	*6*	*48*	$-\infty$	-	-	*conic-bundle*
2	*12*	*16*	*14*	*6*	*15*	*1*	*7*	*-12*	$-\infty$	*63*	*31*	*log-general*
3	*12*	*18*	*21*	*21*	*16*	*0*	*9*	*-102*	*0*	*21*	*21*	*log-general*
4	*12*	*20*	*28*	*36*	*17*	*-1*	*11*	*-192*	*1*	*12*	*20*	*log-general*
5	*12*	*22*	*35*	*51*	*18*	*-2*	*13*	*-282*	*2*	*12*	*22*	*log-general*
6	*12*	*24*	*48*	*96*	*19*	*-5*	*16*	*-456*	*3*	*12*	*24*	*compl. inters.*
7	*12*	*36*	*108*	*324*	*25*	*-19*	*31*	*-1296*	*3*	*12*	*36*	*compl. inters.*
8	*12*	*22*	*23*	*15*	*18*	*1*	*11*	*-138*	$-\infty$	53 48	35 34	*?*

For the cases 1 to 7 we know examples, whereas we cannot decide whether a 3-fold in $\mathbb{P}^5$ with the invariants of case 8 exists.

Uniqueness is known in case 1 [BOSS 2] and for the complete intersections of the cases 6 and 7 respectively. In case 2 we can show that the 2^{nd} reduction has to be a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. In case 3 the 2^{nd} reduction is also uniquely determined. Note that for the cases 4 to 7 the 2^{nd} reduction is isomorphic to $\hat{X}$ and that in case 1 there does not exist a 2^{nd} reduction.

The resolutions of the ideal-sheafs $\mathcal{J}_{\hat{X}/\mathbb{P}^5}$ of the 3-folds in $\mathbb{P}^5$ of degree 12 are the following ones:

case	resolution of $\mathcal{J}_{\hat{X}/\mathbb{P}^5}$
1	$0 \longrightarrow 4\mathcal{O}_{\mathbb{P}^5}(-5) \oplus \Omega^4_{\mathbb{P}^5} \longrightarrow \Omega^2_{\mathbb{P}^5}(-2) \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0$
2	$0 \longrightarrow 2\Omega^2_{\mathbb{P}^5}(-3) \longrightarrow 3\Omega^1_{\mathbb{P}^5}(-4) \oplus 6\mathcal{O}_{\mathbb{P}^5}(-5) \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0$
3	$0 \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-5) \oplus \mathcal{O}_{\mathbb{P}^5}(-6) \longrightarrow \Omega^1_{\mathbb{P}^5}(-3) \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0$
4	$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}^5}(-6) \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-4) \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0$
5	$0 \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-6) \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-5) \oplus \mathcal{O}_{\mathbb{P}^5}(-3) \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0$
6	$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-7) \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-4) \oplus \mathcal{O}_{\mathbb{P}^5}(-3) \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0$
7	$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-8) \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-6) \oplus \mathcal{O}_{\mathbb{P}^5}(-2) \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0$

I would like to thank the *Deutsche Forschungsgemeinschaft (DFG)* for financial support. I am also grateful to professor Michael Schneider for inspiring this work and giving me lots of useful hints, furthermore to professor Frank-Olaf Schreyer and professor Andrew-John Sommese for their valuable support.

1. Notations and Preliminaries

We work over the complex field $\mathbb{C}$. All varieties are projective. By "$3-fold$" we always mean a 3-dimensional complex compact manifold, "*surface*" denotes a 2-dimensional compact complex variety, "*curve*" a 1-dimensional compact complex variety.

The greater part of our notations is standard in algebraic geometry. (cf. e.g. [BSS 1], [BSS 2]) Notations with hat $\hat{}$ are principally reserved for a 3-fold $\hat{X}$ embedded into $\mathbb{P}^5$ by $\hat{L} := \mathcal{O}_{\mathbb{P}^5}(1)|\hat{X}$. By $\hat{S}$ we always mean a generic hyperplane section of $\hat{X}$, which is a smooth surface in $\mathbb{P}^4$, and $\hat{C}$ denotes a generic hyperplane section of $\hat{S}$. So $\hat{C}$ is a smooth curve in $\mathbb{P}^3$. Furthermore we denote by $g(\hat{X})$ the *sectional genus* of $\hat{X}$, by $\chi(\mathcal{O}_{\hat{X}})$, respectively $\chi(\mathcal{O}_{\hat{S}})$ the Euler characteristic of $\hat{X}$, respectively of $\hat{S}$ and by $e(\hat{X})$ the topological Euler characteristic. The Kodaira-dimension is as usual denoted by κ.

For a polarized pair (V, L_V), where V is an n-dimensional manifold and L_V an ample line bundle on V there exists a 1^{st} reduction $(V', L_{V'})$, if $K_V + (n-1)L_V$ is nef and big. Note that for a 3-fold $\hat{X} \subset \mathbb{P}^5$ of degree $\hat{d} \neq 7$ one always has $(\hat{X}, \hat{L}) \simeq (X', L_{X'})$ as polarized pairs whenever a 1^{st} reduction $(X', L_{X'})$ exists and that for $\hat{d} \geq 12$ the line bundle $K_{\hat{X}} + \hat{L}$ is always nef and big with one single well-known exception mentionned in 2.6. If $K_{\hat{X}} + \hat{L}$ is nef and big, $\hat{X}$ is said to be of $log-general$ type. For $\hat{X}$ of log-general type the 1^{st} reduction is always isomorphic to $\hat{X}$ and there always exists a 2^{nd} reduction $(X, \mathcal{K})$ together with a birational morphism $\varphi : \hat{X} \longrightarrow X$, the so-called 2^{nd} reduction map. Note that $\mathcal{K}$ is an ample line bundle on X with $\varphi^*\mathcal{K} = K_{\hat{X}} + \hat{L}$ and $\mathcal{K} = K_X + L_X$ with $L_X := \varphi_*(\hat{L})^{\vee\vee}$. The 2^{nd} reduction map will be examined more closely lateron. Further information about the 1^{st} and the 2^{nd} reduction can also be obtained from [BFS].

On $\hat{X}$ respectively on X we define the *pluridegrees* by

$$\hat{d}_i := (K_{\hat{X}} + \hat{L})^i \hat{L}^{3-i} \quad i = 0, \ldots, 3,$$
$$d_i := (K_X + L_X)^i L_X^{3-i} \quad i = 0, \ldots, 3$$

respectively.

A 3-fold of log-general type satisfies a lot of numerical restrictions so that for a fixed degree $\hat{d}$ one gets a finite number of sets of possible invariants. Putting together all those restrictions which are already known from [BSS 1], [BSS 2] and [BBS] a simple C-programme yields the following list of possible sets of invariants of log-general type 3-folds in $\mathbb{P}^5$ of degree 12. For further details see [E, chap. 1].

Proposition 1.1. *The only possible sets of invariants of log-general type 3-folds in $\mathbb{P}^5$ of degree 12 are:*

case	$\hat{d}_0$	$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_3$	$g(\hat{X})$	$\chi(\mathcal{O}_{\hat{X}})$	$\chi(\mathcal{O}_{\hat{S}})$	$e(\hat{X})$
1	12	12	12	12	13	0	5	24
2	12	16	14	6	15	1	7	-12
3	12	16	20	12	15	0	8	-72
4	12	18	21	21	16	0	9	-102
5	12	18	27	27	16	-1	10	-162
6	12	20	22	6	17	1	10	-108
7	12	20	28	12	17	0	11	-168
8	12	20	28	36	17	-1	11	-192
9	12	22	17	9	18	2	10	-78
10	12	22	23	15	18	1	11	-138
11	12	22	29	21	18	0	12	-198
12	12	22	35	27	18	-1	13	-258
13	12	22	35	51	18	-2	13	-282
14	12	24	48	96	19	-5	16	-456
15	12	36	108	324	25	-19	31	-1296

Note that this procedure can be applied in any degree. The length of the list, however, is rapidly increasing with growing degree.

2. Examples of 3-folds in $\mathbb{P}^5$ of degree 12

There are 2 complete intersection 3-folds in $\mathbb{P}^5$ of degree 12:

Example 2.1. *The complete intersection of type* $(2,6)$ *with the invariants* $\hat{d}_0 = 12$, $\hat{d}_1 = 36$, $\hat{d}_2 = 108$, $\hat{d}_3 = 324$, $g(\hat{X}) = 25$, $\chi(\mathcal{O}_{\hat{X}}) = -19$, $\chi(\mathcal{O}_{\hat{S}}) = 31$, $e(\hat{X}) = -1296$ *is the uniquely determined example with maximal sectional genus.*

Example 2.2. *The complete intersection of type* $(3,4)$ *is the uniquely determined example with submaximal sectional genus. The invariants are* $\hat{d}_0 = 12$, $\hat{d}_1 = 24$, $\hat{d}_2 = 48$, $\hat{d}_3 = 96$, $g(\hat{X}) = 19$, $\chi(\mathcal{O}_{\hat{X}}) = -5$, $\chi(\mathcal{O}_{\hat{S}}) = 16$, $e(\hat{X}) = -456$.

Two further examples can be obtained by liaison-techniques [PS, Prop. 4.1] [O3, 3. Theorem 6] from 3-folds in $\mathbb{P}^5$ with degrees smaller than 12 as explained in [E, chap. 2]. By this method one also gets an explicit resolution of the idealsheaf. Thus for examples obtained by liaison generally all invariants are known. We have

Example 2.3. *Linkage* $V \underset{\sim}{\overset{3,5}{}} \hat{X}$ *where* V *denotes the Segre-embedded 3-fold in* $\mathbb{P}^5$ *of degree 3 with resolution*

$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}^5}(-3) \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-2) \longrightarrow \mathcal{J}_V \longrightarrow 0$$

gives a 3-fold $\hat{X} \subset \mathbb{P}^5$ *with resolution*

$$0 \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-6) \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-5) \oplus \mathcal{O}_{\mathbb{P}^5}(-3) \longrightarrow \mathcal{J}_{\hat{X}} \longrightarrow 0.$$

This example has invariants $\hat{d}_0 = 12$, $\hat{d}_1 = 22$, $\hat{d}_2 = 35$, $\hat{d}_3 = 51$, $g(\hat{X}) = 18$, $\chi(\mathcal{O}_{\hat{X}}) = -2$, $\chi(\mathcal{O}_{\hat{S}}) = 13$, $e(\hat{X}) = -282$. *Looking at the resolution of* $K_{\hat{X}}$ *one can show that* $\kappa(\hat{X}) = 2$.

Example 2.4. *Let V denote the 3-fold in $\mathbb{P}^5$ of degree 8 with resolution*

$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}^5}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-4) \oplus 2\mathcal{O}_{\mathbb{P}^5}(-3) \longrightarrow \mathcal{J}_V \longrightarrow 0.$$

Now linkage $V \overset{4,5}{\sim} \hat{X}$ yields a 3-fold $\hat{X} \subset \mathbb{P}^5$ with resolution

$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}^5}(-6) \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-4) \longrightarrow \mathcal{J}_{\hat{X}} \longrightarrow 0$$

and the invariants $\hat{d}_0 = 12$, $\hat{d}_1 = 20$, $\hat{d}_2 = 28$, $\hat{d}_3 = 36$, $g(\hat{X}) = 17$, $\chi(\mathcal{O}_{\hat{X}}) = -1$, $\chi(\mathcal{O}_{\hat{S}}) = 11$, $e(\hat{X}) = -192$. The Kodaira-dimension can be shown to be 1.

Example 2.5. *Chang* [Ch, p. 107] *has already shown that there exists a Buchsbaum 3-fold in $\mathbb{P}^5$ of degree 12 with resolution*

$$0 \longrightarrow 3\mathcal{O}_{\mathbb{P}^5}(-5) \oplus \mathcal{O}_{\mathbb{P}^5}(-6) \longrightarrow \Omega^1_{\mathbb{P}^5}(-3) \longrightarrow \mathcal{J}_{\hat{X}} \longrightarrow 0$$

and invariants $\hat{d}_0 = 12$, $\hat{d}_1 = 18$, $\hat{d}_2 = 21$, $\hat{d}_3 = 21$, $g(\hat{X}) = 16$, $\chi(\mathcal{O}_{\hat{X}}) = 0$, $\chi(\mathcal{O}_{\hat{S}}) = 11$, $e(\hat{X}) = -102$.

Example 2.6. *In* [BOSS 2] *a further 3-fold in $\mathbb{P}^5$ of degree 12 is described. Its resolution is*

$$0 \longrightarrow 4\mathcal{O}_{\mathbb{P}^5}(-5) \oplus \Omega^4_{\mathbb{P}^5} \longrightarrow \Omega^2_{\mathbb{P}^5}(-2) \longrightarrow \mathcal{J}_{\hat{X}} \longrightarrow 0$$

and it has the invariants $\hat{d}_0 = 12$, $\hat{d}_1 = 16$, $\hat{d}_2 = 8$, $\hat{d}_3 = 0$, $g(\hat{X}) = 15$, $\chi(\mathcal{O}_{\hat{X}}) = 2$, $\chi(\mathcal{O}_{\hat{S}}) = 6$, $e(\hat{X}) = 48$. It is the uniquely determined 3-fold in $\mathbb{P}^5$ with these invariants. We also know that it is a conic-bundle over a $K3$-surface and it is the only 3-fold in $\mathbb{P}^5$ of degree 12 that is not of log-general type.

Now we are going to show that there is a further 3-fold in $\mathbb{P}^5$ of degree 12. This 3-fold was constructed with the help of Schreyer and Popescu using *Macaulay* to show the smoothness.

Proposition 2.7. *There exists a log-general type 3-fold $\hat{X} \subset \mathbb{P}^5$ with $\kappa(\hat{X}) = -\infty$ and invariants $\hat{d}_0 = 12$, $\hat{d}_1 = 16$, $\hat{d}_2 = 14$, $\hat{d}_3 = 6$, $g(\hat{X}) = 15$, $\chi(\mathcal{O}_{\hat{X}}) = 1$, $\chi(\mathcal{O}_{\hat{S}}) = 7$, $e(\hat{X}) = -12$ which is the blowing-up of the Bordiga 3-fold along a smooth curve of degree 15 and genus 10. For every 3-fold in $\mathbb{P}^5$ with these invariants the 2^{nd} reduction has to be a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.*

Proof. We construct $\hat{X} \subset \mathbb{P}^5$ as determinantal locus of a vector bundle homomorphism $\psi : \mathcal{F} \to \mathcal{G}$, i.e.

$$\hat{X} = \{p \in \mathbb{P}^5 : rk\psi(p) < rk(\mathcal{G})\}.$$

We will see below how $\mathcal{F}$ and $\mathcal{G}$ can be chosen. This construction is carried out explicitely with *Macaulay* the complete *Macaulay* programme and results being described in [E, sec. 2.3 and appendix B]. Note that *Macaulay*-computations can only be done over a ring with positive characteristic p with $0 < p \leq 31991$. For general reasons it is enough to do the computations for p maximal in order to obtain the same results (esp. concerning the smoothness) for the case that the field is $\mathbb{C}$. Let $S := \mathbb{Z}[x_0, \ldots, x_5]$ and choose a generic S-module-homomorphism $9S(1) \xrightarrow{m} 2S(2)$ which may be considered to be given by a matrix

$$m = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

We are going to construct m as a representation of the finite cohomology module $M := \oplus_{n \in \mathbb{Z}} H^1(\mathcal{J}_{\hat{X}}(n))$. A free resolution of M is the exact sequence

$$0 \to 3S(-5) \to 16S(-4) \to 33S(-3) \to \begin{matrix} 5S(-1) \\ \oplus \\ 30S(-2) \end{matrix} \to \begin{matrix} 12S \\ \oplus \\ 10S(-1) \end{matrix} \xrightarrow{fm.2} 9S(1) \xrightarrow{m} 2S(2) \to M \to 0$$

Choose a linear morphism $6S \xrightarrow{c} 9S(1)$ by multiplying the 9×12 submatrix of $fm.2$ with a random 12×6 matrix with entries in $\mathbb{Z}$. In the free resolution of the transposed morphism $9S(-1) \xrightarrow{tc} 6S$ we look at the restriction $S(-5) \xrightarrow{b} 9S(-1)$. Now the transposed morphism $9S(1) \xrightarrow{tb} S(5)$ together with $i := tb \circ fm.2$ gives rise to the exact sequence

$$0 \longrightarrow 6S(-5) \longrightarrow im\ fm.2 \xrightarrow{i} S \longrightarrow coker\, i \longrightarrow 0.$$
$$im\ fm.2 = ker\ m$$

The sheafified version

$$0 \longrightarrow 6\mathcal{O}_{\mathbb{P}^5}(-5) \xrightarrow{\psi} ker\ m \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0 \qquad (*)$$
$$\mathcal{F} = 6\mathcal{O}_{\mathbb{P}^5}(-5), \quad \mathcal{G} = ker\ m$$

is a resolution of a 2-codimensional subvariety of $\mathbb{P}^5$ of degree 12 and genus 15. From a free minimal resolution of (∗) we can compute all the invariants of $\hat{X}$. So there only remains to be shown the smoothness of $\hat{X}$. This is carried out by *Macaulay*. One can see that the first 6 entries of the matrix i (i is a 1×12 matrix) describe a scroll. In order to show the smoothness of $\hat{X}$ we look at the Jacobi-matrix of i and take some 2×2-minors in the block of the sextics and the quintics. The ideal of these 2×2-minors together with the equations of the scroll contains $sing(\hat{X})$. It is enough to choose randomly 3 minors in the quintics and in the sextics to show that $sing(\hat{X}) = \emptyset$.
So we have constructed a 3-fold in $\mathbb{P}^5$ with the desired invariants. The birational structure given by the 2^{nd} reduction map can also be examined with *Macaulay* wherefrom the claimed structure is deduced [E, sec. 2.3]. Note that the 2^{nd} reduction $(X, \mathcal{K})$ is the well-known Bordiga 3-fold embedded by $|\mathcal{K}|$ into $\mathbb{P}^5$. This implies especially that $\kappa(\hat{X}) = -\infty$. □

Remark 2.8. *The 3-fold described in 2.7 can also be constructed in a slightly different way if one starts with the cohomology table. This has the advantage that one gets a better manageable resolution of $\mathcal{J}_{\hat{X}}$, namely*

$$0 \longrightarrow 2\Omega^2_{\mathbb{P}^5}(-3) \longrightarrow 3\Omega^1_{\mathbb{P}^5}(-4) \oplus 6\mathcal{O}_{\mathbb{P}^5}(-5) \longrightarrow \mathcal{J}_{\hat{X}/\mathbb{P}^5} \longrightarrow 0.$$

Again, for the smoothness of $\hat{X}$ we need a Macaulay computation.

3. THE 2^{nd} REDUCTION $(X, \mathcal{K})$

Analyzing the 2^{nd} reduction map $\varphi : \hat{X} \longrightarrow X$ for a 3-fold $\hat{X} \subset \mathbb{P}^5$ of log-general type can in certain cases show the non-existence of $\hat{X}$ or, at least, one obtains some birational information about $\hat{X}$. The structure of φ for general 3-folds (without the embedding condition into $\mathbb{P}^5$) has been known explicitely [BFS, 0.2.1] where a list of all possible contractions of divisors to points or curves is given. Divisorial contractions, however, may cause singularities in X which make it difficult to define and compute invariants of X. Fortunately, making systematically use of the embedding-condition of $\hat{X}$ into $\mathbb{P}^5$, most of those divisorial contractions can be excluded. The computations which are rather complicated can be found in [E, chap. 3] and also in [BSS 3]. The result is the following

Theorem 3.1. *Let $\hat{X}$ be a 3-fold in $\mathbb{P}^5$ of log-general type and $\varphi : \hat{X} \longrightarrow X$ the 2^{nd} reduction map. If $\hat{d} \neq 10, 13$ the map φ can only blow down disjoint ruled surfaces $D_i \subset \hat{X}$ to smooth curves $C_i \subset X$ where C_i is isomorphic to the base curve of D_i. In case $\hat{d} = 13$ there may occur in addition contractions of disjoint divisors $D \simeq \mathbb{P}^2$ with normal bundle $\mathcal{N}_{D/\hat{X}} = \mathcal{O}_{\mathbb{P}^2}(-2)$ to points.*

Corollary 3.2. *If in 3.1 $\hat{d} \neq 10, 13$ the 2^{nd} reduction is smooth.*

Next we prove some formulae relating the invariants $\hat{d}_i$ of $\hat{X}$ with the invariants d_i of X. For the rest of this article we always restrict our considerations to the case $\hat{d} \neq 10, 13$.

Lemma 3.3. *Let $\varphi : \hat{X} \longrightarrow X$ be the 2^{nd} reduction map and $D \subset \hat{X}$ a ruled surface blown down by φ to a curve $C \subset X$. Furthermore let ℓ denote a fibre of the ruled surface D and $\mathcal{L}$, $\mathcal{L}'$ line bundles in $PIC(X)$. Then we have:*
i) $(\varphi^*\mathcal{L}.\ell) = 0$ *and* $D\ell = -1$,
ii) $(\varphi^*\mathcal{L})(\varphi^*\mathcal{L}')\, D = 0$, *especially* $(\varphi^*\mathcal{L})^2 D = 0$,
iii) $(\varphi^*\mathcal{L})D^2 = -\mathcal{L}\, C$,
iv) $D^3 = -c_1(\mathcal{N}_{C/X})$,
v) $K_X C = -c_1(\mathcal{N}_{C/X}) - 2 + 2g(C)$
vi) $(\varphi^*\mathcal{L})D = (\mathcal{L}C)\ell$ *in* $H^4(\hat{X}, \mathbb{Z})$.

Proof. These are well-known expressions. See e.g. [M, p. 75]. □

Proposition 3.4. *Let $\varphi : \hat{X} \longrightarrow X$ be the 2^{nd} reduction map contracting the disjoint ruled surfaces $D_a := \cup D_i$ to curves $C_a := \cup C_i$. Then we have*
i) $d_3 = \hat{d}_3$,
ii) $d_2 = \hat{d}_2$,
iii) $d_1 = \hat{d}_1 + \mathcal{K}C_a$, *especially* $d_1 \geq \hat{d}_1$,
iv) $\hat{d}_0 + \hat{d}_1 = d_0 + d_1 - 2\sum_i(\hat{L}^2 D_i + g(C_i) - 1)$.

Proof.
i) $\hat{d}_3 = (K_{\hat{X}} + \hat{L})^3 = (\varphi^*\mathcal{K})^3 = \mathcal{K}^3 = (K_X + L_X)^3 = d_3$.

ii) From [BFS, p. 38] we know a formula for $K_{\hat{X}}$ which under our additional assumptions simplifies to

$$K_{\hat{X}} = \varphi^* K_X + D_a.$$

Now with 3.3 we have

$$\begin{aligned} d_2 &= \mathcal{K}^2 L_X = (\varphi^*\mathcal{K})^2 \varphi^* L_X \\ &= (\varphi^*\mathcal{K})^2(\varphi^*\mathcal{K} - \varphi^* K_X) \\ &= (\varphi^*\mathcal{K})^2\varphi^*\mathcal{K} - (\varphi^*\mathcal{K})^2(K_{\hat{X}} - D_a) \\ &= (K_{\hat{X}} + \hat{L})^2\hat{L} + (K_{\hat{X}} + \hat{L})^2 K_{\hat{X}} - (\varphi^*\mathcal{K})^2 K_{\hat{X}} \\ &= \hat{d}_2. \end{aligned}$$

iii) Is proved in a similar way as *ii*).

iv) Because of $K_{\hat{X}} + \hat{L} = \varphi^*\mathcal{K} = \varphi^*(K_X + L_X)$ and $K_{\hat{X}} = \varphi^* K_X + D_a$ we have

$$\hat{L} = \varphi^* L_X - D_a.$$

For j fixed we get

$$\begin{aligned} \hat{L}^2 D_j &= (\varphi^* L_X - \textstyle\sum D_i)^2 D_j \\ &= (\varphi^* L_X)^2 D_j + (\textstyle\sum D_i)^2 D_j - 2\textstyle\sum(\varphi^* L_X) D_i D_j. \end{aligned}$$

The D_i's being disjoint this simplifies with 3.3 *iii*) to

$$\begin{aligned} \hat{L}^2 D_j &= (\varphi^* L_X)^2 D_j + D_j^3 - 2\varphi^* L_X D_j^2 \\ &= D_j^3 + 2(L_X . C_j). \end{aligned}$$

The following computation now shows the assertion (apply frequently 3.3).

$$\begin{aligned} \hat{d}_0 + \hat{d}_1 &= 2\hat{L}^3 + \hat{L}^2 K_{\hat{X}} \\ &= 2(\varphi^* L_X - \textstyle\sum D_i)^3 + (\varphi^* K_X + \textstyle\sum D_i)(\varphi^* L_X - \textstyle\sum D_i)^2 \\ &= 2(\varphi^* L_X)^3 - 5(\varphi^* L_X)^2(\textstyle\sum D_i) + 4(\varphi^* L_X)(\textstyle\sum D_i)^2 - (\textstyle\sum D_i)^3 \\ &\quad + \varphi^* K_X\,(\varphi^* L_X)^2 - 2\varphi^* K_X \varphi^* L_X(\textstyle\sum D_i) + \varphi^* K_X(\textstyle\sum D_i)^2 \\ &= 2(\varphi^* L_X)^3 + \varphi^* K_X(\varphi^* L_X)^2 - 4\textstyle\sum(L_X C_i) - \textstyle\sum(K_X C_i) - \textstyle\sum(D_i)^3 \\ &= 2L_X^3 + L_X^2 K_X - 2\textstyle\sum(2(L_X C_i) + D_i^3) - \textstyle\sum(K_X C_i) + \textstyle\sum(D_i)^3 \\ &= d_0 + d_1 - 2\textstyle\sum \hat{L}^2 D_i - \textstyle\sum(-c_1(\mathcal{N}_{C_i/X}) - 2 + 2g(C_i)) + \textstyle\sum(D_i)^3 \\ &= d_0 + d_1 - 2\textstyle\sum(\hat{L}^2 D_i + g(C_i) - 1). \quad \square \end{aligned}$$

Corollary 3.5. *From 3.4 iv) we see that the congruence* $d_0 \equiv d_1\,(2)$ *holds because there is always the congruence* $\hat{d}_0 \equiv \hat{d}_1\,(2)$ [BBS, p. 844]. □

Lemma 3.6. *Let* $\varphi : \hat{X} \longrightarrow X$ *be the* 2^{nd} *reduction map and* $D_a := \cup D_i$ *the disjoint union of the ruled surfaces which are contracted to* $C_a := \cup C_i$. *Then there hold the following relations:*

i) $6\mathcal{K}C_a = \sum((\hat{d}_0 - 9)\hat{L}^2 D_i + 2(g(C_i) - 1))$,

ii) $d_0 + d_1 - \hat{d}_0 - \hat{d}_1 = 6\mathcal{K}C_a - (\hat{d}_0 - 11)\hat{L}^2 D_a$,

iii) $d_0 - 5d_1 - \hat{d}_0 + 5\hat{d}_1 = (11 - \hat{d}_0)\hat{L}^2 D_a$,

iv) $e(X) = e(\hat{X}) + 6(d_1 - \hat{d}_1) - (\hat{d}_0 - 9)\hat{L}^2 D_a$.

Proof. Only *i)* needs a longer calculation carried out in [E, 3.2.11], whereas *ii)* follows with 3.4*iv)* out of *i)*, *iii)* is obvious from *ii)* inserting $d_1 = \hat{d}_1 + \mathcal{K}C_a$ and finally *iv)* follows from *i)* using the additivity of the topological Euler characteristic and once again $d_1 = \hat{d}_1 + \mathcal{K}C_a$:

$$\begin{aligned} e(\hat{X}) &= e(\hat{X}\backslash D_a) + e(D_a) \\ &= e(X\backslash C_a) + \sum e(D_i) \\ &= e(X) - \sum(2-2g(C_i)) + \sum(4-4g(C_i)) \\ &= e(X) + 2\sum(1-g(C_i)) \\ &= e(X) + (\hat{d}_0 - 9)\hat{L}^2 D_a - 6\mathcal{K}C_a \\ &= e(X) + (\hat{d}_0 - 9)\hat{L}^2 D_a - 6(d_1 - \hat{d}_1). \quad \square \end{aligned}$$

Lemma 3.7. *On the 2^{nd} reduction $(X, \mathcal{K})$ there is the estimate*

$$d_2^2 \geq d_1 d_3.$$

Proof. Apply the generalized Hodge index theorem [BBS, 0.15] with $M := \mathcal{K}$ and $N := L_X$ for $j = 2$. Note that for $j = 2$ the nef assumption on N is not necessary [E, 1.3.2]. □

Lemma 3.8. *If on the 2^{nd} reduction the line bundle $K_X + 2\mathcal{K}$ is nef we get the inequality*

$$3d_1d_2 + 9d_1d_3 - 9d_2^2 + d_0d_2 - 3d_0d_3 - d_1^2 \leq 0.$$

Proof. We apply the generalized Hodge index theorem [BBS, 0.15] with $M := \mathcal{K}$, $N := K_X + 2\mathcal{K}$ and $j = 1$. □

4. Non-Existence

In some cases short arguments, partly already known from the classification in degrees 9 to 11 exclude the existence of 3-folds in $\mathbb{P}^5$.

Proposition 4.1. *There do not exist 3-folds in $\mathbb{P}^5$ of log-general type with invariants as in cases 3,7,11 or 12 in the list* 1.1 *of possible candidates.*

Proof. Each time there holds $\chi(\mathcal{O}_{\hat{X}}) \leq 0$, thus $p_g(\hat{X}) \geq 1$. So we have $\kappa(\hat{X}) \geq 0$. A 3-fold in $\mathbb{P}^5$ of log-general type with non-negative Kodaira-dimension always fulfills the inequality $\hat{d}_3 \geq 3(\chi(\mathcal{O}_{\hat{S}}) - \chi(\mathcal{O}_{\hat{X}})) - 10$ [BSS 2, Lemma 4.2] which gives a contradiction in our cases. □

Proposition 4.2. *There does not exist a 3-fold in $\mathbb{P}^5$ with invariants as in case 1 in the list* 1.1.

Proof. (cf. [BSS 1, Prop. 3.6])
We consider

$$K_{\hat{X}}|\hat{S}.\hat{L}|\hat{S} = K_{\hat{X}}\hat{L}^2 = \hat{d}_1 - \hat{d}_0 = 0.$$

As $\hat{L}|\hat{S} = L_{\hat{S}}$ is ample, application of the usual Hodge index theorem shows that either $(K_{\hat{X}}|\hat{S})^2 < 0$ or $(K_{\hat{X}}|\hat{S})^2 = 0$ and $K_{\hat{X}}|\hat{S} \equiv 0$. Because of

$$(K_{\hat{X}}|\hat{S})^2 = K_{\hat{S}}^2 + L_{\hat{S}}^2 - 2K_{\hat{S}}L_{\hat{S}} = \hat{d}_2 + \hat{d}_0 - 2\hat{d}_1 = 0$$

we obtain $K_{\hat{X}}|\hat{S} \equiv 0$, whence $K_{\hat{X}} \equiv 0$ because $PIC(\hat{X}) \longrightarrow PIC(\hat{S})$ is injective [F, 7.1.5]. From $\chi(\mathcal{O}_{\hat{X}}) = 0$ we deduce that $p_g(\hat{X}) > 0$ which implies linear equivalence $K_{\hat{X}} \sim 0$. So $p_g(\hat{X}) = 1$ and $h^2(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$. The long exact cohomology sequence of

$$0 \longrightarrow \mathcal{O}_{\hat{X}}(-1) \longrightarrow \mathcal{O}_{\hat{X}} \longrightarrow \mathcal{O}_{\hat{S}} \longrightarrow 0$$

contains the part

$$\begin{array}{l} H^2(\hat{X},\mathcal{O}_{\hat{X}}) \longrightarrow H^2(\hat{S},\mathcal{O}_{\hat{S}}) \longrightarrow H^3(\hat{X},\mathcal{O}_{\hat{X}}(-1)) \longrightarrow H^3(\hat{X},\mathcal{O}_{\hat{X}}) \longrightarrow 0. \\ \qquad \| \\ \qquad 0 \end{array}$$

This yields a contradiction because we have

$$\begin{array}{ll} h^2(\hat{S},\mathcal{O}_{\hat{S}}) & = h^0(\hat{S},K_{\hat{S}}) = \chi(\mathcal{O}_{\hat{S}}) - 1 = 4, \\ h^3(\hat{X},\mathcal{O}_{\hat{X}}(-1)) & = h^0(\hat{X},\hat{L}+K_{\hat{X}}) = h^0(\hat{X},\hat{L}) = h^0(\mathbb{P}^5,\mathcal{O}_{\mathbb{P}^5}(1)) = 6, \\ h^3(\hat{X},\mathcal{O}_{\hat{X}}) & = h^0(\hat{X},K_{\hat{X}}) = 1. \ \square \end{array}$$

Proposition 4.3. *There does not exist a 3-fold in* $\mathbb{P}^5$ *with invariants as in case 5 in the list* 1.1.

Proof. As in this case $\hat{d}_1^2 = \hat{d}_0\hat{d}_2$ holds, we know from [BBS, 1.1.2 p. 834] that there must be the equality $\hat{d}_2^2 = \hat{d}_3\hat{d}_1$ as well which, however, is not the case. □

Proposition 4.4. *There does not exist a 3-fold in* $\mathbb{P}^5$ *with invariants as in case 6 in the list* 1.1.

Proof. As all the intersection numbers $K_{\hat{X}}^3$, $K_{\hat{X}}^2\hat{L}$, $K_{\hat{X}}\hat{L}^2$, $\hat{L}^3$ can be computed from the $\hat{d}_i$*'s* we can apply Riemann-Roch to obtain $\chi(\hat{X}, 2K_{\hat{X}}+\hat{L}) = -1$. This, however, is a contradiction because from the Kawamata-Viehweg vanishing theorem there follows $\chi(\hat{X}, 2K_{\hat{X}} + \hat{L}) = h^0(\hat{X}, 2K_{\hat{X}} + \hat{L})$. □

Now we deal with case 9 in 1.1 which we are going to exclude applying the already announced analysis of the 2^{nd} reduction. The strategy is to suppose that there is a 3-fold in $\mathbb{P}^5$ of log-general type with the invariants $\hat{d}_0 = 12$, $\hat{d}_1 = 22$, $\hat{d}_2 = 17$, $\hat{d}_3 = 9$, $g(\hat{X}) = 18$, $\chi(\mathcal{O}_{\hat{X}}) = 2$, $\chi(\mathcal{O}_{\hat{S}}) = 10$, $e(\hat{X}) = -78$. Then we distinguish the cases

A) On $(X, \mathcal{K})$ the line bundle $K_X + 2\mathcal{K}$ is not nef. This can only occur in very special cases [BFS, Thm. 2.2]

B) On $(X, \mathcal{K})$ the line bundle $K_X + 2\mathcal{K}$ is nef but not big. Again this is only possible for a few well-known pairs [BFS, Thm. 2.3]

C) On $(X, \mathcal{K})$ the line bundle $K_X + 2\mathcal{K}$ is nef and big.

With the help of the formulae of section 3 we compute the invariants d_1 and d_0 on X and get in either case a numerical contradiction. So there cannot exist a 2^{nd} reduction of $\hat{X}$ which is a contradiction to the fact that $\hat{X}$ is of log-general type. Thus this case is excluded.

Lemma 4.5. *If there exists a 3-fold $\hat{X} \subset \mathbb{P}^5$ of log-general type with invariants as in case 9 in* 1.1 *then we have the following additional information:*

i) $\kappa(\hat{X}) = -\infty, \; h^2(\hat{X}, \mathcal{O}_{\hat{X}}) = 1,$

ii) $\kappa(X) = -\infty, \; h^2(X, \mathcal{O}_X) = 1, \; \chi(\mathcal{O}_X) = 2, \; q(X) = 0,$

Proof.

i) If $\kappa(\hat{X}) \geq 0$ one gets a contradiction like in 4.1. So $\kappa(\hat{X}) = -\infty$ and $p_g(\hat{X}) = 0$. This implies $h^2(\hat{X}, \mathcal{O}_{\hat{X}}) = 1$.

ii) Follows from the birationality of φ and the smoothness of X. □

Lemma 4.6. *(Exclusion of A)) There is no 3-fold in $\mathbb{P}^5$ with invariants as in case 9 in* 1.1 *such that on the 2^{nd} reduction the line bundle $K_X + 2\mathcal{K}$ is not nef.*

Proof. The possible pairs for $(X, \mathcal{K})$ in [BFS, Thm. 2.2] can be excluded by easy arguments [E, 4.2.9]. □

Lemma 4.7. *(Exclusion of B)) There is no 3-fold in $\mathbb{P}^5$ with invariants as in case 9 in* 1.1 *such that on the 2^{nd} reduction the line bundle $K_X + 2\mathcal{K}$ is nef but not big.*

Proof. There are 3 possible pairs for $(X, \mathcal{K})$ [BFS, Thm. 2.3] two of which are excluded because they necessarily must have $h^2(X, \mathcal{O}_X) = 0$ contradicting 4.5 *ii)*. The remaining possibility states that $(X, \mathcal{K})$ is a generic $\mathbb{P}^1$-bundle over a normal surface B with $\mathcal{K}|F = \mathcal{O}_{\mathbb{P}^1}(1)$, $L_X|F = \mathcal{O}_{\mathbb{P}^1}(3)$ for a general fibre F. As X is smooth (cf. 3.2) theorem 3.2.1 in [BSW] implies that there are only equidimensional fibres. This means that X is a $\mathbb{P}^1$-bundle [BS, Prop 1.4] and thus the base B is smooth, too.

From $d_1 \geq \hat{d}_1$ (cf. 3.4 *iii)*) and $d_1 \leq d_2^2/d_3 = 32.1..$ (cf. 3.7) and the assumption that $K_X + 2\mathcal{K}$ is not big, which leads to

$$0 = (K + 2\mathcal{K})^3 = (3K_X + 2L_X)^3 = 27d_3 - 27d_2 + 9d_1 - d_0 = -d_0 + 9d_1 - 216,$$

we deduce that only the following pairs (d_0, d_1) are possible:

d_0	72	63	54	45	36	27	18	9	0	−9	−18
d_1	32	31	30	29	28	27	26	25	24	23	22

Because of the additivity of the topological Euler characteristic and 3.6 *i)* and 3.4 *iii)* we obtain, using the notation of paragraph 2:

$$\begin{aligned} e(X) &= e(\hat{X}) + 2\textstyle\sum(g(C_i) - 1) \\ &= e(\hat{X}) + 6\mathcal{K}C_a - \textstyle\sum(\hat{d}_0 - 9)\hat{L}^2 D_i \\ &= e(\hat{X}) + 6(d_1 - \hat{d}_1) - 3\hat{L}^2 D_a \\ &= -210 + 6d_1 - 3\hat{L}^2 D_a. \end{aligned}$$

On the other hand, as $X \longrightarrow B$ is a $\mathbb{P}^1$-bundle we also have

$$\begin{aligned} e(X) &= e(\mathbb{P}^1)\, e(B) \\ &= 2(2 - 2h^{1,0}(B) + 2h^{2,0}(B) + h^{1,1}(B)) \\ &= 2(2 + 2 + h^{1,1}(B)) \\ &\geq 10. \end{aligned}$$

Now for all of the possible pairs (d_0, d_1) the numbers $\hat{L}^2 D_a$ and thus also $e(X)$ can be computed with 3.6*iii*). For each pair (d_0, d_1) we get a contradiction to $e(X) \geq 10$ or to $\hat{L}^2 D_a > 0$. So the case that $(X, \mathcal{K})$ is a $\mathbb{P}^1$-bundle is excluded and the lemma is proved. □

Lemma 4.8. *Let $\hat{X} \subset \mathbb{P}^5$ be a 3-fold with invariants as in case 9 in* 1.1. *Assume that on the 2^{nd} reduction the line bundle $K_X + 2\mathcal{K}$ is nef and big. Then we have $H^0(X, K_X + \mathcal{K}) = 0$ and there can only occur the following combinations of d_0 and d_1:*

d_1	d_0
32	60, 62, 64, 66, 68, 70
31	53, 55, 57, 59, 61
30	46, 48, 50, 52
29	39, 41, 43
28	32, 34
27	25
26	16

Proof. We get $H^0(X, K_X + 2\mathcal{K}) = H^0(X, 2K_X + L_X) = H^0(\hat{X}, 2K_{\hat{X}} + \hat{L})$ where [BFS, 0.2.7] was applied. As $K_{\hat{X}} + \hat{L}$ is nef and big the Kawamata-Viehweg vanishing theorem and the theorem of Riemann-Roch give $h^0(\hat{X}, 2K_{\hat{X}} + \hat{L}) = \chi(2K_{\hat{X}} + \hat{L}) = 0$. As to the values for d_1 and d_0: From 3.7 we have $d_1 \leq d_2^2/d_3 = 32.1..$, from 3.8 we get $132d_1 - d_1^2 - 10d_0 \leq 2601$ and the big-condition for $K_X + 2\mathcal{K}$ reads $-216 + 9d_1 - d_0 > 0$. Combing these 3 inequalities together with the congruence $d_0 \equiv d_1$ (2) (cf. 3.5) gives the stated pairs. □

Lemma 4.9. *(Exclusion of C)) There exists no 3-fold $\hat{X} \subset \mathbb{P}^5$ with invariants as in case 9 in* 1.1 *such that on the 2^{nd} reduction the line bundle $K_X + 2\mathcal{K}$ is nef and big.*

Proof. We apply the Riemann-Roch theorem [BOSS 1, 1.3] to K_X which yields

$$\begin{aligned}\chi(K_X) &= \tfrac{1}{6}K_X^3 - \tfrac{1}{4}K_X^3 + \tfrac{1}{12}(K_X^2 + c_2(X))K_X + \chi(\mathcal{O}_X) \\ &= \tfrac{1}{12}c_2(X)K_X + 2.\end{aligned}$$

Since on the other hand we have $\chi(K_X) = -\chi(\mathcal{O}_X) = -2$ we know the intersection number $c_2(X)K_X = -48$. Now, apply again the Riemann-Roch theorem, this time to $K_X + \mathcal{K}$. Inserting $c_2(X)K_X = -48$ and expressing the intersection numbers in terms of the d_i's one obtains

$$\begin{aligned}\chi(K_X + \mathcal{K}) &= \chi(2K_X + L_X) \\ &= \tfrac{1}{12}\Big(6K_X^3 + 13L_X K_X^2 + 9L_X^2 K_X + 2L_X^3 + 2c_2(X)K_X + c_2(X)L_X\Big) \\ &\quad + \chi(\mathcal{O}_X) \\ &= \tfrac{1}{12}\Big(6d_3 - 5d_2 + d_1 + 2c_2(X)K_X + c_2(X)L_X\Big) + \chi(\mathcal{O}_X) \\ &= \tfrac{1}{12}(-103 + d_1 + c_2(X)L_X).\end{aligned}$$

As $h^0(K_X + \mathcal{K}) = 0$ (cf. 4.8) Kodaira-vanishing gives $0 = h^0(K_X + \mathcal{K}) = \chi(K_X + \mathcal{K})$ which leads to $c_2(X)L_X = 103 - d_1$.

Now, Kawamata-Viehweg vanishing and the Riemann-Roch theorem applied to the line bundle $2K_X + 2\mathcal{K}$ lead to

$$\begin{aligned}\chi(2K_X + 2\mathcal{K}) &= \chi(4K_X + 2L_X) \\ &= \tfrac{1}{12}\Big(84d_3 - 106d_2 + 44d_1 - 6d_0 + 4c_2(X)K_X + 2c_2(X)L_X\Big) \\ &\quad + \chi(\mathcal{O}_X) \\ &= \tfrac{1}{12}(-1008 + 42d_1 - 6d_0)\end{aligned}$$

so that we get the necessary numerical condition

$$-1008 + 42d_1 - 6d_0 = 12h^0(X, 2K_X + 2\mathcal{K}) \geq 0.$$

None of the potential pairs (d_0, d_1) of 4.8 fulfills this condition. So the lemma is proved. $\square$

Whenever on a 3-fold $\hat{X} \subset \mathbb{P}^5$ of log-general type the line bundle $K_{\hat{X}} + \hat{L}$ is not ample, the 2^{nd} reduction $(X, \mathcal{K})$ is not isomorphic to $(\hat{X}, \hat{L})$ and one can analyze the structure of the 2^{nd} reduction map as done above. In [E, sec. 4.3] this is carried out for the case 10 in 1.1 and we can state the following

Proposition 4.10. *A 3-fold $\hat{X} \subset \mathbb{P}^5$ with invariants $\hat{d}_0 = 12$, $\hat{d}_1 = 22$, $\hat{d}_2 = 23$, $\hat{d}_3 = 15$, $g(\hat{X}) = 18$, $\chi(\mathcal{O}_{\hat{X}}) = 1$, $\chi(\mathcal{O}_{\hat{S}}) = 11$, $e(\hat{X}) = -138$ can only exist if on the 2^{nd} reduction $(X, \mathcal{K})$ the line bundle $K_X + 2\mathcal{K}$ is nef and big. There are exactly the following two possibilities:*

a) $K_X + 2\mathcal{K}$ is not ample and the 1^{st} reduction $(X', L_{X'})$ of the pair $(X, \mathcal{K})$ is via the nef-value morphism $\Phi_{K_{X'}+L_{X'}} : X' \longrightarrow B$ a generic quadric (conic) bundle over a normal surface B. The invariants of $(X, \mathcal{K})$ are uniquely determined, namely $d_0 = 53$, $d_1 = 35$, $e(X) = -132$. The 1^{st} reduction map contracts exactly one $\mathbb{P}^2$ and the polarizing bundle $\mathcal{K}$ can be ample and globally generated, yet not very ample.

b) $K_X + \mathcal{K}$ is nef, not big and not ample and $(X, \mathcal{K})$ is via the nef-value morphism $\Phi_{K_X+\mathcal{K}} : X \longrightarrow B$ a generic quadric (conic) bundle over a normal surface B. In this case $(X, \mathcal{K})$ has the invaraints $d_0 = 48$, $d_1 = 34$, $e(X) = -138$. The polarizing bundle $\mathcal{K}$ cannot be globally generated.

Proof. See [E, sec. 4.3]. $\square$

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(Received July 21, 1993;
in revised form October 17, 1993)